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Note on a diffraction–amplification problem

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Abstract
We investigate the solution of the equation $\partial_t \mathcal{E}(x,t) - \mathrm{i}\mathcal{D}\partial_x^2 \mathcal{E}(x,t) = \lambda |S(x,t)|^2 \mathcal{E}(x,t)$, for x in a circle and $S(x,t)$ a Gaussian stochastic field with a covariance of a particular form. It is shown that the coupling λ_c at which $\langle|\mathcal{E}|\rangle$ diverges for $t \geqslant 1$ (in suitable units), is always less or equal for $\mathcal{D} > 0$ than $\mathcal{D} = 0$.

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1. Introduction

In a recent work, Asselah, Dai Pra, Lebowitz and Mounaix (ADLM) [1] analysed the divergence of the average solution to the following diffusion–amplification problem:

$$\begin{cases} \partial_t \mathcal{E}(x,t) - \mathcal{D}\Delta\mathcal{E}(x,t) = \lambda S(x,t)^2 \mathcal{E}(x,t) \\ t \geqslant 0 \quad x \in \Lambda \subset \mathbb{R}^d \quad \text{and} \quad \mathcal{E}(x,0) = 1. \end{cases} \tag{1}$$

Here $\mathcal{D} \geqslant 0$ is the diffusion constant, Λ is a d-dimensional torus, $\lambda > 0$ is a coupling constant to the statistically homogeneous Gaussian driver field $S(x,t)$ with $\langle S(x,t)\rangle = 0$ and $\langle S(x,t)^2\rangle = 1$. They proved that, under some reasonable assumptions on the covariance of S, the average solution of (1) with $D > 0$ diverges at an earlier (or equal) time than when $D = 0$. Put otherwise, fix T such that $\langle \mathcal{E}(x,T)\rangle = \infty$ for $\lambda > \lambda_c$ and $\langle \mathcal{E}(x,T)\rangle < \infty$ for $\lambda < \lambda_c$. Then λ_c is smaller than (or equal to) $\overline{\lambda}_c$, the value of λ at which such a divergence occurs for $\mathcal{D} = 0$. ADLM conjectured that this result should also apply to the case where $\mathcal{D}$ is replaced by $\mathrm{i}\mathcal{D}$, i.e. where diffusion is replaced by diffraction, the case of physical interest considered by Rose and DuBois in [2].

The difficulty in proving the above conjecture lies in controlling the complex Feynman path integral, compared to that of the Feynman–Kac formula for the diffusive case. One cannot *a priori* exclude the possibility that destructive interference effects between different paths make the sum of divergent contributions finite, raising the value of the coupling constant at which the average amplification diverges. To understand this diffraction-induced interference between paths, we investigate here the diffraction case in a one–dimensional model ($d = 1$) in which the Gaussian driver field S has a special form specified in section 2. We prove in section 3 that $\langle|\mathcal{E}(x, T)|\rangle = \infty$ for $\lambda > \lambda_c$ with $\lambda_c \leqslant \overline{\lambda}_c$. Possible generalizations are discussed in section 4.

2. Model and definitions

We consider the diffraction–amplification equation

$$\begin{cases} \partial_t \mathcal{E}(x,t) - \frac{\mathrm{i}}{2}\Delta\mathcal{E}(x,t) = \lambda|S(x,t)|^2\mathcal{E}(x,t) \\ x \in \Lambda_1 \quad \text{and} \quad \mathcal{E}(x,0) = 1 \end{cases} \tag{2}$$

where $\lambda > 0$ is the coupling constant and Λ_1 is a circle of unit circumference. The case in which the circle has circumference L and/or there is a constant $\mathcal{D}$ multiplying $\Delta\mathcal{E}$ is straightforwardly obtained by rescaling x, t and λ. The driver amplitude $S(x, t)$ is a spacetime homogeneous complex Gaussian random field with

$$\begin{cases} \langle S(x,t)\rangle = \langle S(x,t)S(x',t')\rangle = 0 \\ \langle S(x,t)S^*(x',t')\rangle = C(x-x',t-t') \end{cases} \tag{3}$$

and $C(0, 0) = 1$. We can write $S(x, t)$ in the form

$$S(x,t) = \sum_{n\in\mathbb{Z}} \xi_n(t)\,\mathrm{e}^{2\mathrm{i}\pi nx} \tag{4}$$

with $\xi_n(t)$ Gaussian random functions satisfying

$$\begin{cases} \langle \xi_n(t)\rangle = \langle \xi_n(t)\xi_m(t')\rangle = 0 \\ \langle \xi_n(t)\xi_m^*(t')\rangle = \delta_{nm}C_n(t-t') \end{cases} \tag{5}$$

with $C_n(0) \equiv \epsilon_n \geqslant 0$ and $\sum \epsilon_n = 1$. We now assume that only a finite number of ϵ_n are non-vanishing

$$\epsilon_n = 0 \qquad \text{for} \quad |n| > N \quad N < \infty \tag{6}$$

reducing the right-hand side (rhs) of equation (4) to a finite sum of $M = 2N + 1$ terms, from $n = -N$ to $n = N$. We further assume that

$$\xi_n(t) = \sqrt{\epsilon_n}\phi_n(t)s_n \tag{7}$$

where the $\phi_n(t)$ are specified functions of t and the s_n are independent complex Gaussian random variables with

$$\begin{cases} \langle s_n\rangle = \langle s_n s_m\rangle = 0 \\ \langle s_n s_m^*\rangle = \delta_{nm}. \end{cases} \tag{8}$$

It then follows from (5), (7) and (8) that

$$\phi_n(t) = \exp(\mathrm{i}\omega_n t) \qquad \omega_n \text{ real} \tag{9}$$

yielding

$$C(x-x',t-t') = \sum_{n=-N}^{N} \epsilon_n\,\mathrm{e}^{\mathrm{i}[2\pi n(x-x')+\omega_n(t-t')]}. \tag{10}$$

In the following we take $\omega_n = an^2$, $a > 0$, which is the case of interest in optics where the spacetime behaviour of $C(x, t)$ corresponds to a diffraction along x as t increases. The last and most restrictive assumption we make is that the $\phi_{n\geqslant 0}(t)$ are orthogonal functions of t in [0, 1], which specifies a. One finds

$$\omega_n = 2\pi n^2 \qquad \text{i.e.} \quad \phi_n(t) = \exp(2\mathrm{i}\pi n^2 t). \tag{11}$$

Equation (2) can thus be rewritten as

$$\partial_t \mathcal{E}(x,t) - \frac{\mathrm{i}}{2}\Delta\mathcal{E}(x,t) = \lambda s^\dagger \gamma(x,t) s \mathcal{E}(x,t) \tag{12}$$

where s is the M-line Gaussian random vector the elements of which are the s_n, and $\gamma(x,t)$ is an $M \times M$ Hermitian matrix with elements

$$\gamma_{nm}(x,t) = \sqrt{\epsilon_n\epsilon_m}\, \mathrm{e}^{-2\mathrm{i}\pi[(n-m)x+(n^2-m^2)t]}. \tag{13}$$

Finally, the critical coupling λ_c and its diffraction-free counterpart $\overline{\lambda}_c$ are defined by

$$\lambda_c = \inf\{\lambda > 0 : \langle|\mathcal{E}(0,1)|\rangle = +\infty\} \tag{14}$$

$$\overline{\lambda}_c = \inf\{\lambda > 0 : \langle \mathrm{e}^{\lambda\int_0^1 S(0,t)^2\mathrm{d}t}\rangle = +\infty\} \tag{15}$$

where $\langle\cdot\rangle$ denotes the average over the realizations of S. Equations (14) and (15) give the values of λ at which $\langle|\mathcal{E}(x,t)|\rangle$ diverges after one unit of time with and without diffraction respectively.

3. Comparison of λ_c and $\overline{\lambda}_c$

We begin with two lemmas that will be useful in the following. Let $\mathcal{E}_\gamma(x,t)$ be the solution to equation (12) for a given realization of s.

Lemma 1. *For every $x \in \mathbb{R}$ and $t \in [0,1]$, and every $M \times M$ unitary matrix P, one has $\langle|\mathcal{E}_\gamma(x,t)|\rangle = \langle|\mathcal{E}_{P^\dagger\gamma P}(x,t)|\rangle$.*

Proof. Let $B(x,t)$ be the set of all the continuous paths $x(\tau)$, with $t \in [0,1]$, $\tau \leqslant t$, and $x(\tau) \in \mathbb{R}$, arriving at $x(t) = x$. Writing the solution to equation (12) as a Feynman path integral, one has

$$\begin{aligned}
\langle|\mathcal{E}_\gamma(x,t)|\rangle &= \int_{\mathbb{C}^M} \frac{\mathrm{e}^{-|s|^2}}{\pi^M} \left| \int_{x(\cdot)\in B(x,t)} \exp\left(\int_0^t \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda s^\dagger\gamma(x(\tau),\tau)s\right]\mathrm{d}\tau\right)\right.\\
&\qquad \left.\times\,\mathrm{d}[x(\cdot)]\right| \prod_n \mathrm{d}^2 s_n\\
&= \int_{\mathbb{C}^M} \frac{\mathrm{e}^{-s^\dagger PP^\dagger s}}{\pi^M} \left| \int_{x(\cdot)\in B(x,t)} \exp\left(\int_0^t \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda s^\dagger PP^\dagger\gamma(x(\tau),\tau)PP^\dagger s\right]\mathrm{d}\tau\right)\right.\\
&\qquad \left.\times\,\mathrm{d}[x(\cdot)]\right| \prod_n \mathrm{d}^2 s_n\\
&= \int_{\mathbb{C}^M} \frac{\mathrm{e}^{-|\sigma|^2}}{\pi^M} \left| \int_{x(\cdot)\in B(x,t)} \exp\left(\int_0^t \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda \sigma^\dagger P^\dagger\gamma(x(\tau),\tau)P\sigma\right]\mathrm{d}\tau\right)\right.\\
&\qquad \left.\times\,\mathrm{d}[x(\cdot)]\right| \prod_n \mathrm{d}^2 \sigma_n = \langle|\mathcal{E}_{P^\dagger\gamma P}(x,t)|\rangle.
\end{aligned}$$

Here we have used $PP^\dagger = 1$ and made the change of variables $s_n \to \sigma_n$, where the σ_n are the components of $\sigma \equiv P^\dagger s$. Note that lemma 1 applies also to the diffraction-free case by eliminating the path integral and setting $x(\tau) \equiv x$. □

Let κ_n ($n \in \mathbb{N}$) be the eigenvalues of the $M \times M$ Hermitian matrix $\int_0^1 \gamma(0,t)\,\mathrm{d}t$. One has the following lemma:

Lemma 2. $\overline{\lambda}_c = (\sup_n \kappa_n)^{-1}$.

Proof. Using equation (13) one finds, after a suitable permutation of lines and columns, that $\int_0^1 \gamma(0,t)\,\mathrm{d}t$ can be written in the block-diagonal form

$$\int_0^1 \gamma(0,t)\,\mathrm{d}t = \begin{pmatrix} \epsilon_0 & 0 & \cdots & & & \\ 0 & g_1 & 0 & \cdots & & \\ \vdots & 0 & \ddots & 0 & \cdots & \\ & \vdots & 0 & g_{N-1} & 0 & \\ & & \vdots & 0 & g_N & \end{pmatrix} \tag{16}$$

with

$$g_j = \begin{pmatrix} \epsilon_j & \sqrt{\epsilon_j \epsilon_{-j}} \\ \sqrt{\epsilon_j \epsilon_{-j}} & \epsilon_{-j} \end{pmatrix} \tag{17}$$

the diagonalization of which yields the M eigenvalues κ_n. These eigenvalues are easily found to be ϵ_0, $\epsilon_j + \epsilon_{-j}$ and 0. The matrix diagonalizing (16), P, is a unitary matrix given by

$$P = \begin{pmatrix} 1 & 0 & \cdots & & & \\ 0 & p_1 & 0 & \cdots & & \\ \vdots & 0 & \ddots & 0 & \cdots & \\ & \vdots & 0 & p_{N-1} & 0 & \\ & & \vdots & 0 & p_N & \end{pmatrix} \tag{18}$$

with

$$p_j = \begin{pmatrix} \sqrt{\epsilon_j/(\epsilon_j + \epsilon_{-j})} & \sqrt{\epsilon_{-j}/(\epsilon_j + \epsilon_{-j})} \\ \sqrt{\epsilon_{-j}/(\epsilon_j + \epsilon_{-j})} & -\sqrt{\epsilon_j/(\epsilon_j + \epsilon_{-j})} \end{pmatrix}. \tag{19}$$

Using the diffraction-free version of lemma 1 with P given by equations (18) and (19), one obtains

$$\begin{aligned} \left\langle \mathrm{e}^{\lambda \int_0^1 S(0,t)^2 \mathrm{d}t} \right\rangle &= \int_{\mathbb{C}^M} \frac{\mathrm{e}^{-|\sigma|^2}}{\pi^M}\, \mathrm{e}^{\lambda \sigma^\dagger [\int_0^1 P^\dagger \gamma(0,t) P\,\mathrm{d}t]\sigma} \prod_n \mathrm{d}^2\sigma_n \\ &= \prod_n \int_0^{+\infty} \mathrm{e}^{(\lambda\kappa_n - 1)u_n}\,\mathrm{d}u_n \end{aligned} \tag{20}$$

with $u_n \equiv |\sigma_n|^2$, from which lemma 2 follows straightforwardly. One can now prove the proposition: □

Proposition. $\lambda_c \leqslant \overline{\lambda}_c$.

Proof. From lemma 1 with P given by equations (18) and (19), one has

$$\begin{aligned} \langle |\mathcal{E}(0,1)| \rangle = \int_{\mathbb{C}^M} \frac{\mathrm{e}^{-|\sigma|^2}}{\pi^M} \Bigg| \int_{x(\cdot)\in B(0,1)} \exp\left(\int_0^1 \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda \sigma^\dagger P^\dagger \gamma(x(\tau),\tau) P\sigma \right] \mathrm{d}\tau \right) \\ \times \mathrm{d}[x(\cdot)] \Bigg| \prod_n \mathrm{d}^2\sigma_n. \end{aligned} \tag{21}$$

For this integral to exist it is necessary that

$$\lim_{|\sigma|\to+\infty} \mathrm{e}^{-|\sigma|^2} \left| \int_{x(\cdot)\in B(0,1)} \exp\left(\int_0^1 \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda\sigma^\dagger P^\dagger \gamma(x(\tau),\tau) P\sigma \right] \mathrm{d}\tau \right) \mathrm{d}[x(\cdot)] \right| = 0 \tag{22}$$

for all the directions $\sigma/|\sigma|$ in $\mathbb{C}^M$. We will now show that this cannot happen for $\lambda \geqslant \overline{\lambda}_c$. Let $\kappa_m = \sup_n \kappa_n$. From lemma 2 one has $\kappa_m = 1/\overline{\lambda}_c$. Now, consider equation (22) for $\sigma_n = 0, n \neq m$, and $\sigma_m = z \in \mathbb{C}$. One finds after some straightforward algebra

$$\sigma^\dagger P^\dagger \gamma(x,t) P\sigma = \left[\frac{1}{\overline{\lambda}_c} - \alpha_m \overline{\lambda}_c \sin^2(2\pi k x) \right] |z|^2 \tag{23}$$

and

$$\begin{aligned} &\mathrm{e}^{-|\sigma|^2} \left| \int_{x(\cdot)\in B(0,1)} \exp\left(\int_0^1 \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda\sigma^\dagger P^\dagger \gamma(x(\tau),\tau) P\sigma \right] \mathrm{d}\tau \right) \mathrm{d}[x(\cdot)] \right| \\ &\quad = \mathrm{e}^{(\lambda/\overline{\lambda}_c - 1)|z|^2} \left| \int_{x(\cdot)\in B(0,1)} \exp\left(\int_0^1 \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 - \lambda|z|^2\alpha_m\overline{\lambda}_c \sin^2\left(2\pi k x(\tau)\right) \right] \mathrm{d}\tau \right) \right. \\ &\qquad \left. \times \mathrm{d}[x(\cdot)] \right| \end{aligned} \tag{24}$$

where $\alpha_m = 4\epsilon_k\epsilon_{-k}$ if $\kappa_m = \epsilon_k + \epsilon_{-k}$, which defines k, and $\alpha_m = 0$ if $\kappa_m = \epsilon_0$. There are two possibilities:

(i) If $\alpha_m = 0$ one has

$$\begin{aligned} &\mathrm{e}^{-|\sigma|^2} \left| \int_{x(\cdot)\in B(0,1)} \exp\left(\int_0^1 \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda\sigma^\dagger P^\dagger \gamma(x(\tau),\tau) P\sigma \right] \mathrm{d}\tau \right) \mathrm{d}[x(\cdot)] \right| \\ &\quad = \mathrm{e}^{(\lambda/\overline{\lambda}_c - 1)|z|^2} \left| \int_{x(\cdot)\in B(0,1)} \exp\left(\int_0^1 \frac{\mathrm{i}}{2}\dot{x}(\tau)^2\, \mathrm{d}\tau \right) \mathrm{d}[x(\cdot)] \right| = \mathrm{e}^{(\lambda/\overline{\lambda}_c - 1)|z|^2}. \end{aligned} \tag{25}$$

If $\lambda_c > \overline{\lambda}_c$ this expression diverges as $|z|$ tends to infinity, which is in contradiction with equation (22).

(ii) If $\alpha_m \neq 0$ the leading term of the asymptotic expansion of the path integral (24) in the large $|z|$ limit is given by the contribution of the paths near $x(\tau) = 0$. Expanding $\sin^2(2\pi k x)$ around $x = 0$ at the lowest order and performing the resulting Gaussian integral, one obtains the asymptotics

$$\begin{aligned} &\mathrm{e}^{-|\sigma|^2} \left| \int_{x(\cdot)\in B(0,1)} \exp\left(\int_0^1 \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 + \lambda\sigma^\dagger P^\dagger \gamma(x(\tau),\tau) P\sigma \right] \mathrm{d}\tau \right) \mathrm{d}[x(\cdot)] \right| \\ &\quad \sim \sqrt{2}\mathrm{e}^{(\lambda/\overline{\lambda}_c - 1)|z|^2} \exp(-|z|\pi k\sqrt{\alpha_m\lambda\overline{\lambda}_c}) \qquad (|z| \to +\infty). \end{aligned} \tag{26}$$

Again, if $\lambda_c > \overline{\lambda}_c$ the rhs of this expression diverges as $|z|$ tends to infinity, which completes the proof of the proposition. □

4. Discussion and perspectives

As a conclusion we would like to outline a possible way of fitting the ideas behind this calculation to a more general proof of the conjecture. First, it should be noticed that what makes the proof here possible is the slow decrease of the asymptotic behaviour of the path integral on the rhs of equation (24) as $|z| \to +\infty$. Namely, denoting by $f(|z|)$ this path integral, one has $\forall\varepsilon > 0$, $\lim_{|z|\to+\infty} |f(|z|)| \exp(\varepsilon|z|^2) = +\infty$ (cf equations (25) and (26)), which proves the conjecture by leading to a contradiction with equation (22).

Now, consider the case in which $S(x,t)$ is given by a finite Karhunen–Loève-type expansion $S(x,t) = \sum_n s_n \Phi_n(x,t)$ with $x \in \mathbb{R}^d$, $t \in [0,T]$, and $\Phi_n(x,t)$ not necessarily periodic in time[3]. With such an expression for $S(x,t)$ on the rhs of equation (1), one finds that the equation for $\mathcal{E}(x,t)$ takes on the same form as in (12) with $\gamma_{nm}(x,t) = \Phi_n(x,t)\Phi_m(x,t)^*$. One can now systematically replace, from equation (21) on, the matrix diagonalizing $\int_0^1 \gamma(0,t)\,\mathrm{d}t$ by the one diagonalizing $\Gamma[y(\cdot)] \equiv \int_0^T \gamma(y(t),t)\,\mathrm{d}t$, where $y(\cdot) \in B(0,T)$ is a continuous path maximizing the largest eigenvalue of $\Gamma[x(\cdot)]$.[4] Denoting by κ_c this maximized largest eigenvalue, one expects the rhs of equation (24) to be replaced by

$$\mathrm{e}^{(\lambda\kappa_c-1)|z|^2} \left| \int_{x(\cdot)\in B(0,T)} \exp\left(\int_0^T \left[\frac{\mathrm{i}}{2}\dot{x}(\tau)^2 - \lambda|z|^2 V(x(\tau),\tau) \right] \mathrm{d}\tau \right) \mathrm{d}[x(\cdot)] \right| \tag{27}$$

where $V(x,t)$ is a real potential given by some linear combination of the $\gamma_{nm}(x,t)$ and such that

$$\inf_{x(\cdot)\in B(0,T)} \int_0^T V(x(\tau),\tau)\,\mathrm{d}\tau = 0. \tag{28}$$

The proof would then proceed along exactly the same lines as in this note: denote by $f(|z|)$ the path integral in equation (27), if one can prove that $\forall\varepsilon > 0$, $\lim_{|z|\to+\infty} |f(|z|)| \exp(\varepsilon|z|^2) > 0$ (which seems to be the difficult part of the matter), then we will have proved $\lambda_c \leqslant \kappa_c^{-1}$. Finally, since $1/\overline{\lambda}_c$ is the largest eigenvalue of $\Gamma[x(\cdot) = 0]$, it is necessarily smaller than (or equal to) κ_c, and $\lambda_c \leqslant \kappa_c^{-1}$ implies $\lambda_c \leqslant \overline{\lambda}_c$. Note that equation (24) is a particular case of equations (27) and (28) with $\kappa_c = 1/\overline{\lambda}_c$ and $V(x,t) = \alpha_m \overline{\lambda}_c \sin^2(2\pi kx)$.

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[3] Note that (4) belongs to this class of driver with $d = 1$ and $\Phi_n(x,t) = \sqrt{\epsilon_n}\exp[2\mathrm{i}\pi(nx + n^2 t)]$.

[4] In the cases where there is no such path, one should consider a path which realizes the supremum up to an arbitrarily small constant.